Journal of Approximation Theory **117**, 189–206 (2002)
doi:10.1006/jath.2002.3696

Oblique Projections and Abstract Splines

G. Corach[1,2]

Instituto Argentino de Matemática, Saavedra 15 Piso 3 (1083), Buenos Aires, Argentina and Departamento de Matematica, Facultad de Ingenieria, Paseo Colon 850, Buenos Aires, Argentina
E-mail : gcorach@dm.uba.ar

A. Maestripieri

Instituto de Ciencias, UNGS, Roca 850 (1663) San Miguel, Argentina
E-mail : amaestri@ungs.edu.ar

and

D. Stojanoff[3]

Departamento de Matemática, FCE-UNLP, 115 y 50 (1900) La Plata, Argentina
E-mail : demetrio@mate.unlp.edu.ar

Communicated by Frank Deutsch

Received October 31, 2000; accepted in revised form April 1, 2002

Given a closed subspace $\mathcal{S}$ of a Hilbert space $\mathcal{H}$ and a bounded linear operator $A \in L(\mathcal{H})$ which is positive, consider the set of all A-self-adjoint projections onto $\mathcal{S}$:

$$\mathcal{P}(A, \mathcal{S}) = \{Q \in L(\mathcal{H}) : Q^2 = Q, \quad Q(\mathcal{H}) = \mathcal{S},\ AQ = Q^*A\}.$$

In addition, if $\mathcal{H}_1$ is another Hilbert space, $T : \mathcal{H} \to \mathcal{H}_1$ is a bounded linear operator such that $T^*T = A$ and $\xi \in \mathcal{H}$, consider the set of $(T, \mathcal{S})$ spline interpolants to ξ:

$$sp(T, \mathcal{S}, \xi) = \left\{\eta \in \xi + \mathcal{S} : \|T\eta\| = \min_{\sigma \in \mathcal{S}} \|T(\xi + \sigma)\|\right\}.$$

A strong relationship exists between $\mathcal{P}(A, \mathcal{S})$ and $sp(T, \mathcal{S}, \xi)$. In fact, $\mathcal{P}(A, \mathcal{S})$ is not empty if and only if $sp(T, \mathcal{S}, \xi)$ is not empty for every $\xi \in \mathcal{H}$. In this case, for any $\xi \in \mathcal{H} \setminus \mathcal{S}$ it holds

$$sp(T, \mathcal{S}, \xi) = \{(1 - Q)\xi : Q \in \mathcal{P}(A, \mathcal{S})\}$$

[1] To whom correspondence should be addressed.
[2] Partially supported by CONICET (PIP 4463/96), Universidad de Buenos Aires (UBACYT TX92 and TW49).
[3] Partially supported by CONICET (PIP 4463/96), Universidad de Buenos Aires (UBACYT TW49).

0021-9045/02 $35.00

and for any $\xi \in \mathscr{H}$, the unique vector of $sp(T, \mathscr{S}, \xi)$ with minimal norm is $(1 - P_{A,\mathscr{S}})\xi$, where $P_{A,\mathscr{S}}$ is a distinguished element of $\mathscr{P}(A, \mathscr{S})$. These results offer a generalization to arbitrary operators of several theorems by de Boor, Atteia, Sard and others, which hold for closed range operators.

1. INTRODUCTION

Given two Hilbert spaces $\mathscr{H}$ and $\mathscr{H}_1$, $T \in L(\mathscr{H}, \mathscr{H}_1)$, $\mathscr{S} \subseteq \mathscr{H}$ a closed subspace and $\xi \in \mathscr{H}$, an abstract spline or a $(T, \mathscr{S})$-spline interpolant to ξ is any element of the set

$$sp(T, \mathscr{S}, \xi) = \left\{ \eta \in \xi + \mathscr{S} : \|T\eta\| = \min_{\sigma \in \mathscr{S}} \|T(\xi + \sigma)\| \right\}.$$

Observe that $A = T^*T = |T|^2$, as a positive bounded operator on $\mathscr{H}$, defines a semiinner product $\langle \cdot, \cdot \rangle_A : \mathscr{H} \times \mathscr{H} \to \mathbb{C}$ by $\langle \xi, \eta \rangle_A = \langle A\xi, \eta \rangle$, $\xi, \eta \in \mathscr{H}$ and a corresponding seminorm $\| \cdot \|_A : \mathscr{H} \to \mathbb{R}^+$ given by $\|\eta\|_A = \langle \eta, \eta \rangle_A^{1/2} = \langle A\eta, \eta \rangle^{1/2} = \|T\eta\|$. Thus, if for any $\eta \in \mathscr{H}$ we consider $d_A(\eta, \mathscr{S}) = \inf_{\sigma \in \mathscr{S}} \|\eta + \sigma\|_A$, then

$$sp(T, \mathscr{S}, \xi) = \{ \eta \in \xi + \mathscr{S};\ \|\eta\|_A = d_A(\xi, \mathscr{S}) \}.$$

If A is an invertible operator, then $\langle , \rangle_A$ is a scalar product, $(\mathscr{H}, \langle , \rangle_A)$ is a Hilbert space and, by the projection theorem, $d_A(\xi, \mathscr{S}) = \|(I - P_{A,\mathscr{S}})\xi\|_A$ and $sp(T, \mathscr{S}, \xi) = \{(I - P_{A,\mathscr{S}})\xi\}$, where $P_{A,\mathscr{S}}$ is unique orthogonal projection onto $\mathscr{S}$ which is orthogonal to the inner product $\langle , \rangle_A$. However, if A is not invertible then $\| \cdot \|_A$ is or a seminorm or an incomplete norm and we cannot use the projection theorem unless we complete the quotient $\mathscr{H}/\ker A$. One of the main goals of this paper is to get a simpler way of describing the set $sp(T, \mathscr{S}, \xi)$.

We start with a positive bounded linear operator A on a Hilbert space $\mathscr{H}$ and a closed subspace $\mathscr{S}$ of $\mathscr{H}$. The subspace $\mathscr{S}^{\perp_A} = \{\xi : \langle A\xi, \eta \rangle = 0\ \forall \eta \in \mathscr{S}\}$ is called the *A-orthogonal companion* of $\mathscr{S}$. Note the identities

$$\mathscr{S}^{\perp_A} = A^{-1}(\mathscr{S}^{\perp}) = A(\mathscr{S})^{\perp} = \ker(PA). \tag{1}$$

Instead of defining adjoint operators with respect to $\langle , \rangle_A$, we restrict our discussion to A-self-adjoint operators, i.e. $W \in L(\mathscr{H})$ such that $AW = W^*A$. Note that any such W satisfies $\langle W\xi, \eta \rangle_A = \langle \xi, W\eta \rangle_A$, $\xi, \eta \in \mathscr{H}$.

The pair $(A, \mathscr{S})$ is said to be *compatible* if there exists a projection $Q \in L(\mathscr{H})$ such that $Q(\mathscr{H}) = \mathscr{S}$ and $AQ = Q^*A$. The main result in this paper is the description of the relationship between the set

$$\mathscr{P}(A, \mathscr{S}) = \{ Q \in \mathscr{Q} : R(Q) = \mathscr{S},\ AQ = Q^*A \}$$

and $sp(T, \mathscr{S}, \xi)$, where $T : \mathscr{H} \to \mathscr{H}_1$ is any bounded linear operator such that $T^*T = A$. A relevant point here is that this method allows to tackle the case of operators with non-closed range. Thus, several results by Atteia [3], Sard [18], Golomb [11], Shekhtman [19], de Boor [4], Izumino [13], Delvos [9], Deutsch [8] are generalized to any bounded linear operators T.

If $(A, \mathscr{S})$, is compatible, there exists a distinguished element $P_{A,\mathscr{S}} \in \mathscr{P}(A, \mathscr{S})$. The study of the map $(A, \mathscr{S}) \to P_{A,\mathscr{S}}$ was initiated by Pasternak-Winiarski [15] at least for invertible A. A geometrical description of that map can be found in [2]. In [7, 12] the inversibility hypothesis on A was removed, opening, in that way, the possibility that $\mathscr{P}(A, \mathscr{S})$ be empty or have many elements. This induces the notion of compatibility of a pair $(A, \mathscr{S})$. This paper is mainly devoted to explore the relationship of the compatibility of $(A, \mathscr{S})$ with the existence of spline interpolants for every $\xi \in \mathscr{H}$. Section 2 contains a short study on compatibility of a pair $(A, \mathscr{S})$. If $(A, \mathscr{S})$ is compatible, the properties of the distinguished element $P_{A,\mathscr{S}} \in \mathscr{P}(A, \mathscr{S})$ are described. In Section 3, we show that $(A, \mathscr{S})$ is compatible if and only if $sp(T, \mathscr{S}, \xi)$ is not empty for any $\xi \in \mathscr{H}$ and that $sp(T, \mathscr{S}, \xi) = \{(1 - Q)\xi : Q \in \mathscr{P}(A, \mathscr{S})\}$ for any $\xi \in \mathscr{H} \setminus \mathscr{S}$. Moreover, the vector of $sp(T, \mathscr{S}, \xi)$ with minimal norm is exactly $(1 - P_{A,\mathscr{S}})\xi$. In Section 4, we present some characterizations of $P_{A,\mathscr{S}}$ which are useful for the study of the convergence of $\{P_{A,\mathscr{S}_n}\xi\}$ if $(A, \mathscr{S}_n)$ is compatible for every $n \in \mathbb{N}$ and $\mathscr{S}_n$ decreases to 0. This study is the goal of Section 5. Finally, Section 6 includes several examples of compatibility and spline projections.

In this paper, $L(\mathscr{H})$ is the algebra of all linear bounded operators on the Hilbert space $\mathscr{H}$ and $L(\mathscr{H})^+$ is the subset of $L(\mathscr{H})$ of all self-adjoint positive (i.e., non-negative definite) operators. For every $C \in L(\mathscr{H})$ its range is denoted by $R(C)$. If $R(C)$ is closed, then $C^\dagger$ denotes the Moore–Penrose pseudoinverse of C. The orthogonal projections onto a closed subspace $\mathscr{S}$ is denoted by $P_{\mathscr{S}}$. The direct sum of subspaces $\mathscr{S}$ and $\mathscr{T}$ is denoted $\mathscr{S} \dot{+} \mathscr{T}$. Finally, $\mathscr{S} \ominus \mathscr{T}$ denotes $\mathscr{S} \cap \mathscr{T}^\perp$.

2. A-SELF-ADJOINT PROJECTIONS

Throughout this paper $\mathscr{S}$ denotes a closed subspace of $\mathscr{H}$ and A is a fixed operator in $L(\mathscr{H})^+$. Recall that $\mathscr{S}^{\perp_A} = A^{-1}(\mathscr{S}^\perp)$. It is easy to see that a projection Q belongs to $\mathscr{P}(A, \mathscr{S})$ if and only if $R(Q) = \mathscr{S}$ and $ker\, Q \subseteq A^{-1}(\mathscr{S}^\perp)$. Then

$$\text{the pair } (A, \mathscr{S}) \text{ is compatible if and only if } \mathscr{S} + A^{-1}(\mathscr{S}^\perp) = \mathscr{H}. \tag{2}$$

In this case, $\mathscr{P}(A, \mathscr{S})$ has a single element if and only if $ker\, A \cap \mathscr{S} = \{0\}$ because

$$\mathscr{S} \cap A^{-1}(\mathscr{S}^\perp) = ker\, A \cap \mathscr{S}. \tag{3}$$

If $(A, \mathcal{S})$ is compatible , then there is a distinguished element in $\mathcal{P}(A, \mathcal{S})$, namely the unique projection $P_{A,\mathcal{S}}$ onto $\mathcal{S}$ with kernel $A^{-1}(\mathcal{S}^{\perp}) \ominus (\ker A \cap \mathcal{S})$. The elements of $\mathcal{P}(A, \mathcal{S})$ can be parametrized by the set of relative supplements of $\ker A \cap \mathcal{S}$ into $A^{-1}(\mathcal{S}^{\perp})$.

The set $\mathcal{P}(A, \mathcal{S})$ can also be characterized using the matrix operator decomposition induced by the orthogonal projection $P = P_{\mathcal{S}}$. Under this representation, A has a matrix form

$$A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}, \tag{4}$$

where $a \in L(\mathcal{S})^+$, $b \in L(\mathcal{S}^{\perp}, \mathcal{S})$ and $c \in L(\mathcal{S}^{\perp})^+$. Observe that $P = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $PA = \left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)$ and $PAP = \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Every projection Q with range $\mathcal{S}$ has the matrix form $Q = \left(\begin{smallmatrix} 1 & x \\ 0 & 0 \end{smallmatrix}\right)$ for some $x \in L(\mathcal{S}^{\perp}, \mathcal{S})$. It is easy to see that $Q \in \mathcal{P}(A, \mathcal{S})$ if and only if x satisfies the equation $ax = b$. Then

$$\mathcal{P}(A, \mathcal{S}) = \left\{ Q = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} : x \in L(\mathcal{S}^{\perp}, \mathcal{S}) \text{ and } ax = b \right\}. \tag{5}$$

Note that Eq. (5) implies that if $(A, \mathcal{S})$ is compatible, then $R(b) \subseteq R(a)$. As a corollary of a well-known theorem of R.G. Douglas, it can be shown that these two conditions are, indeed, equivalent. First, we recall Douglas' theorem [10]:

THEOREM 2.1. *Let $B, C \in L(\mathcal{H})$. Then the following conditions are equivalent*:

1. $R(B) \subseteq R(C)$.
2. *There exists a positive number λ such that $BB^* \leqslant \lambda CC^*$.*
3. *There exists $D \in L(\mathcal{H})$ such that $B = CD$.Moreover, there exists a unique operator D which satisfies the conditions*

$$B = CD, \quad \ker D = \ker B \quad \text{and} \quad R(D) \subseteq \overline{R(C^*)}.$$

In this case, $\|D\|^2 = \inf\{\lambda : BB^* \leqslant \lambda CC^*\}$; D is called the reduced *solution of the equation $CX = B$. If $R(C)$ is closed, then $D = C^{\dagger}B$.*

COROLLARY 2.2. *Let $A \in L(\mathcal{H})^+$ and $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace. If A has matrix form as in* (4), *then $(A, \mathcal{S})$ is compatible if and only if $R(b) \subseteq R(a)$.*

The next theorem describes some properties of $\mathscr{P}(A,\mathscr{S})$ and $P_{A,\mathscr{S}}$. The norm of $P_{A,\mathscr{S}}$ will be computed in Section 5.

THEOREM 2.3. *Let $A \in L(\mathscr{H})^+$ with matrix form* (4), *such that the pair $(A,\mathscr{S})$ is compatible.*

1. *The distinguished projection $P_{A,\mathscr{S}} \in \mathscr{P}(A,\mathscr{S})$ has the matrix form*

$$P_{A,\mathscr{S}} = \begin{pmatrix} 1 & d \\ 0 & 0 \end{pmatrix},$$

where $d \in L(\mathscr{S}^\perp,\mathscr{S})$ is the reduced solution of the equation $ax = b$.

2. *$\mathscr{P}(A,\mathscr{S})$ is an affine manifold which can be parametrized as*

$$\mathscr{P}(A,\mathscr{S}) = P_{A,\mathscr{S}} + L(\mathscr{S}^\perp,\mathscr{N}),$$

where $\mathscr{N} = A^{-1}(S^\perp) \cap \mathscr{S} = \ker A \cap \mathscr{S}$ and $L(\mathscr{S}^\perp,\mathscr{N})$ is viewed as a subspace of $L(\mathscr{H})$. A matrix representation of this parametrization is

$$\mathscr{P}(A,\mathscr{S})Q = P_{A,\mathscr{S}} + z = \begin{array}{c} \begin{pmatrix} 1 & 0 & d \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{l} \mathscr{S}\ominus\mathscr{N} \\ \mathscr{N} \\ \mathscr{S}^\perp \end{array}. \tag{6}$$

3. *$P_{A,\mathscr{S}}$ has minimal norm in $\mathscr{P}(A,\mathscr{S})$, i.e.* $\|P_{A,\mathscr{S}}\| = \min\{\|Q\|\colon Q \in \mathscr{P}(A,\mathscr{S})\}$.

Proof.

(1) If $Q = \begin{pmatrix} 1 & d \\ 0 & 0 \end{pmatrix}$, then $Q \in \mathscr{P}(A,\mathscr{S})$ and $\ker Q \subseteq A^{-1}(\mathscr{S}^\perp)$. Since $P_{A,\mathscr{S}}$ is characterized by the properties $R(P_{A,\mathscr{S}}) = \mathscr{S}$ and $\ker P_{A,\mathscr{S}} = A^{-1}(\mathscr{S}^\perp) \ominus \mathscr{N}$ then, in order to show that $Q = P_{A,\mathscr{S}}$ it suffices to prove that $\ker Q \subseteq \mathscr{N}^\perp$. Let $\xi \in \ker Q$ and write $\xi = \xi_1 + \xi_2$ with $\xi_1 \in \mathscr{S}$ and $\xi_2 \in \mathscr{S}^\perp$. Then $0 = Q\xi = \xi_1 + d\xi_2$. If $\eta \in \mathscr{N}$, then $\langle \xi,\eta\rangle = \langle \xi_1,\eta\rangle = -\langle d\xi_2,\eta\rangle = 0$ because, by Theorem 2.1, $R(d) \subseteq \overline{R(a)}$ and, as an operator in $L(\mathscr{S})$, $\ker a = \mathscr{S} \cap \ker PAP = \mathscr{S} \cap \ker A = \mathscr{N}$.

(2) Let $Q = \begin{pmatrix} 1 & y \\ 0 & 0 \end{pmatrix}$ with $y \in L(\mathscr{S}^\perp,\mathscr{S})$ and let $d \in L(\mathscr{S}^\perp,\mathscr{S})$ be the reduced solution of the equation $ax = b$. Then $Q \in \mathscr{P}(A,\mathscr{S})$ if and only if $ay = b$. Therefore, if $z = y - d$, then $Q \in \mathscr{P}(A,\mathscr{S})$ if and only if $Q = P_{A,\mathscr{S}} + z$ and $R(z) \subseteq \ker a = \mathscr{N}$. Concerning the matrix representation (6), recall that $R(d) \subseteq \overline{R(a)} = (\ker a)^\perp = \mathscr{S} \ominus \mathscr{N}$. Therefore,

$$Q = P_{A,\mathscr{S}} + z = \begin{pmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \mathscr{S}\ominus\mathscr{N} \\ \mathscr{N} \\ \mathscr{S}^\perp \end{array}.$$

(3) If $Q \in \mathcal{P}(A, \mathcal{S})$ has the matrix form given in Eq. (6), then

$$\|Q\|^2 = \|QQ^*\| = 1 + \left\| \begin{pmatrix} 0 & 0 & d \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \right\|^2 \geqslant 1 + \|d\|^2 = \|P_{A,\mathcal{S}}\|^2. \quad \blacksquare$$

Remark 2.4. Under additional hypothesis on A, other characterizations of compatibility can be used. We mention a sample of these, taken from [6, 7]:

1. If A is injective then the following conditions are equivalent: (a)The pair $(A, \mathcal{S})$ is compatible. (b)$\mathcal{S}^\perp \subseteq R(A + \lambda(1 - P))$ for some (and then for any) $\lambda > 0$. (c)$P(\overline{A(\mathcal{S})}) = \mathcal{S}$ and $\overline{A(\mathcal{S})} \cap \mathcal{S}^\perp = \{0\}$.
2. If A has closed range then the following conditions are equivalent: (a)The pair $(A, \mathcal{S})$ is compatible. (b)$R(PAP)$ is closed. (c)$\mathcal{S} + \ker A$ is closed.
3. If $R(PAP)$ is closed (or, equivalently, if $R(PA^{1/2})$ or $A^{1/2}(\mathcal{S})$ are closed), then $(A, \mathcal{S})$ is compatible. Indeed, using the matrix form (4), the positivity of A implies that $R(b) \subseteq R(a^{1/2})$ (see, e.g., [1]). If $R(PAP) = R(a)$ is closed, then $R(b) \subseteq R(a^{1/2}) = R(a)$ so that $(A, \mathcal{S})$ is compatible by Corollary 2.2.

3. SPLINES AND A-SELF-ADJOINT PROJECTIONS

In this section, we characterize the existence of splines in terms of the existence of A-self-adjoint projections. The first result extends a theorem of Izumino [13] to operators whose ranges are not necessarily closed.

PROPOSITION 3.1. *Let $T \in L(\mathcal{H}, \mathcal{H}_1)$, $A = T^*T \in L(\mathcal{H})$ and $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace. Then, for any $\xi \in \mathcal{H}$,*

$$sp(T, \mathcal{S}, \xi) = (\xi + \mathcal{S}) \cap \mathcal{S}^{\perp_A}.$$

In particular, $sp\ (T, \mathcal{S}, \xi)$ is an affine manifold of $L(\mathcal{H})$ and, if $\eta \in sp$, $(T\mathcal{S}, \xi)$, then $sp\ (T, \mathcal{S}, \xi) = \eta + \ker T \cap \mathcal{S}$.

Proof. Suppose that $\eta \in (\xi + \mathcal{S}) \cap A^{-1}(\mathcal{S}^\perp)$ and $\sigma \in \mathcal{S}$. Then $\langle A\eta, \sigma \rangle = \langle A\sigma, \eta \rangle = 0$ and

$$\|T(\eta + \sigma)\|^2 = \langle A(\eta + \sigma), \eta + \sigma \rangle = \langle A\eta, \eta \rangle + \langle A\sigma, \sigma \rangle \geqslant \langle A\eta, \eta \rangle = \|T\eta\|^2.$$

Therefore, $\eta \in sp\ (T, \mathcal{S}, \xi)$. Conversely, if $\eta \in sp\ (T, \mathcal{S}, \xi)$ and $\sigma \in \mathcal{S}$, then, for any $t \in \mathbb{R}$,

$$\begin{aligned} \|T\eta\|^2 \leqslant \|T(\eta + t\sigma)\|^2 &= \langle A(\eta + t\sigma), \eta + t\sigma \rangle \\ &= \langle A\eta, \eta \rangle + t^2 \langle A\sigma, \sigma \rangle + 2t \operatorname{Re}\langle A\eta, \sigma \rangle \\ &= \|T\eta\|^2 + t^2 \langle A\sigma, \sigma \rangle + 2t \operatorname{Re}\langle A\eta, \sigma \rangle, \end{aligned}$$

therefore $t^2 \langle A\sigma, \sigma \rangle + 2t \operatorname{Re}\langle A\eta, \sigma \rangle \geqslant 0$ for all $t \in \mathbb{R}$ and a standard argument shows that $\langle A\eta, \sigma \rangle = 0$ and then $\eta \in (\xi + \mathcal{S}) \cap A^{-1}(\mathcal{S}^{\perp})$. ∎

THEOREM 3.2. *Let $T \in L(\mathcal{H}, \mathcal{H}_1)$, $A = T^*T \in L(\mathcal{H})$ and $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace.*

1. *If $\xi \in \mathcal{H}$, $sp\ (T, \mathcal{S}, \xi)$ is not empty $\xi \in \mathcal{S} + A^{-1}(\mathcal{S}^{\perp})$.*
2. *The following conditions are equivalent*: (a)*$sp\ (T, \mathcal{S}, \xi)$ is not empty for every $\xi \in \mathcal{H}$.* (b)*$\mathcal{S} + A^{-1}(\mathcal{S}^{\perp}) = \mathcal{H}$.* (c)*The pair $(A, \mathcal{S})$ is compatible.*
3. *If $(A, \mathcal{S})$ is compatible and $\xi \in \mathcal{H} \setminus \mathcal{S}$, it holds $sp\ (T, \mathcal{S}, \xi) = \{(I - Q)\xi : Q \in \mathcal{P}(A, \mathcal{S})\}$.*
4. *If $(A, \mathcal{S})$ is compatible, then for every $\xi \in \mathcal{H}$, $(I - P_{A,\mathcal{S}})\xi$ is the unique vector in $sp\ (T, \mathcal{S}, \xi)$ with minimal norm.*

Proof. The first assertion follows directly from Proposition 3.1. Indeed, if $\eta \in sp\ (T, \mathcal{S}, \xi)$ and $\eta = \xi + \sigma$ with $\sigma \in \mathcal{S}$, then $\xi = -\sigma + \eta \in \mathcal{S} + A^{-1}(\mathcal{S}^{\perp})$; the converse implication is similar. The second assertion follows from the first one and Eq. (2). In order to prove the third item, let $\xi \in \mathcal{H}$ and $Q \in \mathcal{P}(A, \mathcal{S})$. Then, by Proposition 3.1 and Eq. (2),

$$(I - Q)\xi = \xi - Q\xi \in (\xi + \mathcal{S}) \cap \ker Q \subseteq (\xi + \mathcal{S}) \cap A^{-1}(\mathcal{S}^{\perp}) = sp\ (T, \mathcal{S}, \xi).$$

Conversely, let $\eta \in sp\ (T, \mathcal{S}, \xi)$ and $\sigma \in \mathcal{S}$ such that $\xi = \sigma + \eta$. We are looking for some $Q \in \mathcal{P}(A, \mathcal{S})$ such that $Q\xi = \sigma$. Let $\eta_1 = (I - P_{A,\mathcal{S}})\xi$ and $\sigma_1 = \xi - \eta_1 = P_{A,\mathcal{S}}\xi \in \mathcal{S}$. Then, by Proposition 3.1,

$$\sigma - \sigma_1 = \eta_1 - \eta \in \mathcal{S} \cap A^{-1}(\mathcal{S}^{\perp}) = \ker A \cap \mathcal{S}.$$

If $\xi = \sigma_2 + \rho$ with $\sigma_2 \in \mathcal{S}$ and $0 \neq \rho \in \mathcal{S}^{\perp}$, choose $z \in L(\mathcal{S}^{\perp}, \ker A \cap \mathcal{S})$ ($\subseteq L(\mathcal{H})$) such that $z(\rho) = \sigma - \sigma_1$. By Theorem 2.3, $Q = P_{A,\mathcal{S}} + z \in \mathcal{P}(A, \mathcal{S})$ and clearly $Q\xi = \sigma$.

The minimality of $\|(1 - P_{A,\mathcal{S}})\xi\|$ is proved as follows. If $\xi \in \mathcal{S}$, then $(I - P_{A,\mathcal{S}})\xi = 0$, which must be minimal. If $\xi \notin \mathcal{S}$, let $\xi = \sigma_2 + \rho$ with $\sigma_2 \in \mathcal{S}$ and $0 \neq \rho \in \mathcal{S}^{\perp}$. By Theorem 2.3, any $Q \in \mathcal{P}(A, \mathcal{S})$ has the form $Q = P_{A,\mathcal{S}} + z$, with $z \in L(\mathcal{S}^{\perp}, \ker A \cap \mathcal{S})$ ($\subseteq L(\mathcal{H})$). Recall that $R(P_{A,\mathcal{S}}) = \mathcal{S} \ominus (\ker A \cap$

$\mathcal{S}$). Therefore,

$$\|(I-Q)\xi\|^2 = \|(I-Q)\rho\|^2 = \|\rho - P_{A,\mathcal{S}}(\rho) - z(\rho)\|^2 = \|\rho\|^2 + \|P_{A,\mathcal{S}}(\rho)\|^2 + \|z(\rho)\|^2$$

$$\geqslant \|\rho\|^2 + \|P_{A,\mathcal{S}}(\rho)\|^2 = \|\rho - P_{A,\mathcal{S}}(\rho)\|^2 = \|(I-P_{A,\mathcal{S}})\xi\|^2. \quad \blacksquare$$

COROLLARY 3.3. *Let* $T \in L(\mathcal{H}, \mathcal{H}_1)$, $A = T^*T \in L(\mathcal{H})$ *and* $\mathcal{S} \subseteq \mathcal{H}$ *a closed subspace. Then the following are equivalent*:

1. *sp* $(T, \mathcal{S}, \xi)$ *has a unique element for every* $\xi \in \mathcal{H}$.
2. *The pair* $(A, \mathcal{S})$ *is compatible and ker* $T \cap \mathcal{S} = \{0\}$.

Remark 3.4. Let $T \in L(\mathcal{H}, \mathcal{H}_1)$, $A = T^*T \in L(\mathcal{H})$ and $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace.

1. If $(A, \mathcal{S})$ is compatible then, by item 4 of Theorem 3.2, the projection $1 - P_{A,\mathcal{S}}$ coincides with the so-called *spline projection* for T and $\mathcal{S}$ when T has a closed range.
2. If $R(T)$ is closed, then, by Remark 2.4 and Theorem 3.2, *sp* $(T, \mathcal{S}, \xi) \neq \emptyset$ for every $\xi \in \mathcal{H}$ if and only if *ker* $T + \mathcal{S}$ is closed. In case that *ker* $T \cap \mathcal{S} = \{0\}$, then it is equivalent to the condition that the inclination between *ker* T and $\mathcal{S}$ is less than one (see [4, 8]).
3. If $\xi \in \mathcal{S}$, then *sp* $(T, \mathcal{S}, \xi) =$ *ker* $T \cap \mathcal{S}$. On the other hand, $(I-Q)\xi = 0$ for every $Q \in \mathcal{P}(A, \mathcal{S})$. So the equality of item 3 of Theorem 3.2 may be false in this case.

4. CHARACTERIZATIONS OF THE SPLINE PROJECTION $P_{A,\mathcal{S}}$

Fix $A \in L(\mathcal{H})^+$ and a closed subspace $\mathcal{S} \subseteq \mathcal{H}$. As before, we denote $P = P_{\mathcal{S}}$. In this section, two different descriptions of the spline projection $P_{A,\mathcal{S}}$ are given and, as a consequence, we relate $P_{A,\mathcal{S}}$ with the shorted operator (see [1] and Remark 4.4 below).

By Corollary 2.2, it holds that the pair $(A, \mathcal{S})$ is compatible if and only if $R(PA) \subseteq R(PAP)$. In case that A is invertible, it is known (see [2]) that, in the matrix form (4), a is invertible in $L(\mathcal{S})$ and

$$P_{A,\mathcal{S}} = \begin{pmatrix} a^{-1} & 0 \\ 0 & 0 \end{pmatrix} \qquad PA = \begin{pmatrix} 1 & a^{-1}b \\ 0 & 0 \end{pmatrix} \tag{7}$$

because $a^{-1}b$ is the reduced solution of $ax = b$ (see Theorem 2.3). Rewriting (7), we get $(PAP)P_{A,\mathcal{S}} = PA$. Thus, if A is invertible, $P_{A,\mathcal{S}}$ is the reduced solution of the equation $(PAP)X = PA$. Let us consider the general case, in other words, if the pair $(A, \mathcal{S})$ is compatible, let us relate $P_{A,\mathcal{S}}$ with the reduced solution Q of the equation

$$(PAP)X = PA. \tag{8}$$

Observe that, in general, $\overline{R(PAP)}$ is strictly contained in $\mathcal{S}$. Therefore, $R(Q)$ may be smaller that $\mathcal{S} = R(P_{A,\mathcal{S}})$.

PROPOSITION 4.1. *If the pair* $(A, \mathcal{S})$ *is compatible*, Q *is the reduced solution of* Eq. (8) *and* $\mathcal{N} = \ker A \cap \mathcal{S}$, *then*

$$P_{A,\mathcal{S}} = P_{\mathcal{N}} + Q.$$

Moreover, Q *verifies the following properties*:

1. $Q^2 = Q$, $\ker Q = A^{-1}(\mathcal{S}^{\perp})$ *and* $R(Q) = \mathcal{S} \ominus \mathcal{N}$.
2. Q is A-*self-adjoint*.
3. $Q = P_{A,\mathcal{S}\ominus\mathcal{N}}$.

Proof. Using the matrix form (4) of A, observe that, in $L(\mathcal{S})$, $\ker a = \mathcal{N}$ and $\overline{R(a)} = \overline{R(a^{1/2})} = \mathcal{S} \ominus \mathcal{N}$. Note that $R(Q) \subseteq \overline{R(a)}$. Also $\ker Q = \ker(PA)$ $= A^{-1}(\mathcal{S}^{\perp})$. If $\xi \in \mathcal{S} \ominus \mathcal{N}$, then

$$a(Q\xi) = (PAP)Q\xi = PA\xi = PAP\xi = a(\xi).$$

Since a is injective in $\mathcal{S} \ominus \mathcal{N}$, we can deduce that $Q\xi = \xi$ for all $\xi \in \mathcal{S} \ominus \mathcal{N}$. Now, the compatibility of $(A, \mathcal{S})$ implies that $\mathcal{S} + A^{-1}(\mathcal{S}^{\perp}) = \mathcal{H}$. Also $A^{-1}(\mathcal{S}^{\perp}) \cap \mathcal{S} = \ker A \cap \mathcal{S} = \mathcal{N}$. Therefore $A^{-1}(\mathcal{S}^{\perp}) \dot{+} (\mathcal{S} \ominus \mathcal{N}) = \mathcal{H}$. Then $Q^2 = Q$ and $R(Q) = \mathcal{S} \ominus \mathcal{N}$. Note that

$$\ker Q = A^{-1}(\mathcal{S}^{\perp}) \subseteq A^{-1}((\mathcal{S} \ominus \mathcal{N})^{\perp}) = R(Q)^{\perp_A},$$

so that Q is A-self-adjoint by Eq. (2). On the other hand, $(\mathcal{S} \ominus \mathcal{N}) \cap \ker A = \{0\}$, so that Q is the unique element of $P(A, \mathcal{S} \ominus \mathcal{N})$, by Theorem 2.3. Observe that $R(Q) \subseteq \mathcal{N}^{\perp}$ and $\mathcal{N} \subseteq \ker A \subseteq A^{-1}(\mathcal{S}^{\perp}) = \ker Q$. Therefore, $(P_{\mathcal{N}} + Q)^2 = P_{\mathcal{N}} + Q$, $R(P_{\mathcal{N}} + Q) = \mathcal{S}$ and $\ker(P_{\mathcal{N}} + Q) = (A^{-1}(\mathcal{S}^{\perp}))$ $\ominus \mathcal{N}$. These formulae clearly imply that $P_{\mathcal{N}} + Q = P_{A,\mathcal{S}}$ (see Theorem 2.3). ∎

PROPOSITION 4.2. *If* $(A, \mathcal{S})$ *is compatible and* $\mathcal{M} = \overline{A^{1/2}(\mathcal{S})}$, *then* $R(P_{\mathcal{M}}A^{1/2}) \subseteq R(A^{1/2}P)$. *Moreover*, Eq. (8) *and*

$$(A^{1/2}P)X = P_{\mathcal{M}}A^{1/2} \tag{9}$$

have the same reduced solution. In particular, if $A^{1/2}(\mathcal{S})$ is closed and ker $A \cap \mathcal{S} = \{0\}$, then

$$P_{A,\mathcal{S}} = (A^{1/2}P)^{\dagger}P_{\mathcal{M}}A^{1/2} = (A^{1/2}P)^{\dagger}A^{1/2} = (TP)^{\dagger}T \tag{10}$$

for every $T \in L(\mathcal{H}, \mathcal{H}_1)$ such that $T^*T = A$.

Proof. Denote $B = A^{1/2}$. Recall that $\mathcal{M} = \overline{B(\mathcal{S})} = B^{-1}(\mathcal{S}^{\perp})^{\perp}$. Observe that

$$BP_{\mathcal{M}}B = AP_{A,\mathcal{S}} = APP_{A,\mathcal{S}} : \tag{11}$$

in fact, for $\xi \in \mathcal{H}$, let $\eta = P_{A,\mathcal{S}}\xi$ and $\rho = \xi - \eta \in A^{-1}(\mathcal{S}^{\perp})$; then $B\eta \in \mathcal{M}$ and $B\rho \in B^{-1}(\mathcal{S}^{\perp}) = \mathcal{M}^{\perp}$. Hence, $BP_{\mathcal{M}}B\xi = A\eta = AP_{A,\mathcal{S}}\xi$. By Proposition 4.1, the projection $Q = P_{A,\mathcal{S}} - P_{\mathcal{N}}$ is the reduced solution of the equation $PAP\,X = PA$. We shall see that Q is the reduced solution of Eq. (9). First note that, by Eq. (11), $BP_{\mathcal{M}}B = (AP)P_{A,\mathcal{S}} = (AP)Q$, so $B(P_{\mathcal{M}}B - BPQ) = 0$. But $R(P_{\mathcal{M}}B - BPQ) \subseteq \overline{R(B)} = (\ker B)^{\perp}$. Hence, Q is a solution of (9). Note that $\ker P_{\mathcal{M}}B = B^{-1}(B^{-1}(\mathcal{S}^{\perp})) = A^{-1}(\mathcal{S}^{\perp}) = \ker Q$ by Proposition 4.1. Finally,

$$\overline{R((BP)^*)} = \overline{R(PB)} = \overline{R(PAP)} = \mathcal{S} \ominus \mathcal{N} = R(Q).$$

The first equality of Eq. (10) follows directly. The second, from the fact that $(A^{1/2}P)^{\dagger}P_{\mathcal{M}} = (A^{1/2}P)^{\dagger}$. The last equality follows easily using the polar decomposition of T because $A^{1/2} = |T|$. ∎

Formula (10), for operators with closed range, is due to Golomb [11].

COROLLARY 4.3. *Under the notations of Proposition* 4.2, *the pair* $(A, \mathcal{S})$ *is compatible if and only if* $R(P_{\mathcal{M}}A^{1/2}) \subseteq R(A^{1/2}P)$.

Proof. Suppose that $R(P_{\mathcal{M}}A^{1/2}) \subseteq R(A^{1/2}P)$. Then, given $\xi \in \mathcal{H}$, there must exist $\sigma \in \mathcal{S}$ such that $P_{\mathcal{M}}A^{1/2}\xi = A^{1/2}\sigma$. Therefore, $A^{1/2}(\xi - \sigma) = (1 - P_{\mathcal{M}})A^{1/2}\xi$ and

$$\begin{aligned} \|A^{1/2}(\xi - \sigma)\| &= \|(1 - P_{\mathcal{M}})A^{1/2}\xi\| = d(A^{1/2}\xi, A^{1/2}(\mathcal{S})) \\ &= \inf\{\|A^{1/2}(\xi + \tau)\| : \tau \in \mathcal{S}\}. \end{aligned} \tag{12}$$

Hence, $\xi - \sigma \in sp\,(T, \mathcal{S}, \xi)$ and $sp\,(T, \mathcal{S}, \xi) \neq \emptyset$ for every $\xi \in \mathcal{H}$. This implies compatibility by Theorem 3.2. The converse implication was shown in Proposition 4.2. ∎

Remark 4.4. If $A \in L(\mathcal{H})^+$ and $\mathcal{S} \in \mathcal{H}$ is a closed subspace, then the set

$$\{X \in L(\mathcal{H})^+ : X \leqslant A \text{ and } R(X) \subseteq \mathcal{S}^\perp\}$$

has a maximum (for the natural order relation in $L(\mathcal{H})^+$), which is called the *shorted operator* of A to $\mathcal{S}^\perp$. We denote it by $\Sigma(P, A)$. This notion, due to Krein [14] and Anderson–Trapp [1], has many applications to electrical engineering. It is well known (see [16]) that

$$\Sigma(P, A) = A^{1/2} P_{\mathcal{T}} A^{1/2},$$

where $\mathcal{T} = A^{-1/2}(\mathcal{S}^\perp) = A^{1/2}(\mathcal{S})^\perp$. From the proof of Proposition 4.2, it follows that, if $(A, \mathcal{S})$ is compatible, then $A^{1/2}(1 - P_{\mathcal{T}})A^{1/2} = AP_{A,\mathcal{S}}$. Therefore, in this case, $\Sigma(P, A) = A(1 - P_{A,\mathcal{S}})$. More generally, it can be shown that $\Sigma(P, A) = A(1 - Q)$ for every $Q \in \mathcal{P}(A, \mathcal{S})$ (see [7]).

5. CONVERGENCE OF SPLINE PROJECTIONS

This section is devoted to the study of the convergence of abstract splines in the general (i.e. not necessarily closed range) case. Given $A \in L(\mathcal{H})^+$, let us consider a sequence of closed subspaces $\mathcal{S}_n$ such that all pairs $(A, \mathcal{S}_n)$ are compatible. Following de Boor [4] and Izumino [13], it is natural to look for conditions which are equivalent to the fact that $P_{A,\mathcal{S}_n} \to^{SOT} 0$ (i.e. the spline projections converge to I), where $\to^{SOT}$ means convergence in the strong operator topology. This problem has a well-known solution under the assumption that $R(A)$ is closed (see [4] or [13]). However, in our more general setting, it is possible that the sequence $\{\mathcal{S}_n\}$ decreases to $\{0\}$, while $\|P_{A,\mathcal{S}_n}\|$ tends to infinity (see Example 5.7). This induces us to consider the following weaker convergence:

DEFINITION 5.1. Let $A \in L(\mathcal{H})^+$ and $T_n,\ T \in L(\mathcal{H})$, $n \in \mathbb{N}$. We shall say that the sequence T_n converges A-SOT to T: $T_n \to^{A-SOT} T$ if

$$\|(T_n - T)\xi\|_A \to 0 \quad \text{for every } \xi \in \mathcal{H}.$$

Note that $T_n \to^{A-SOT} T$ if and only if $A^{1/2}T_n \to^{SOT} A^{1/2}T$.

We start with the computation of the norm of $P_{A,\mathcal{S}}$ for any compatible pair $(A, \mathcal{S})$. Before that, recall the following formula, due to Ptak [17] (see also [5, 7]): if Q_1 and Q_2 are orthogonal projections such that $R(Q_1)+R(Q_2) = \mathcal{H}$, then the norm of the unique projection Q_3 with $ker\, Q_3 = R(Q_1)$

and $R(Q_3) = R(Q_2)$ is

$$\|Q_3\| = (1 - \|Q_1 Q_2\|^2)^{-1/2}. \tag{13}$$

PROPOSITION 5.2. *Let $A \in L(\mathscr{H})^+$ such that the pair $(A, \mathscr{S})$ is compatible. Then,*

$$\|P_{A,\mathscr{S}}\|^2 = \inf\{\lambda > 0: \ PA^2P \leqslant \lambda (PAP)^2\}. \tag{14}$$

If, in addition, $\ker A \cap \mathscr{S} = \{0\}$, then

$$\|P_{A,\mathscr{S}}\| = (1 - \|QP\|^2)^{-1/2}, \tag{15}$$

where Q denotes the orthogonal projection onto $A^{-1}(\mathscr{S}^\perp)$.

Proof. Let Q be the reduced solution of the equation $(PAP)X = PA$. Then $\|Q\|^2$ equals the infimum of Eq. (14) by Douglas Theorem. On the other hand, by Proposition 4.1, $\|Q\| = \|P_{A,\mathscr{S}}\|$, showing formula (14). If $\ker A \cap \mathscr{S} = \{0\}$, then Theorem 2.3 assures that $R(P_{A,\mathscr{S}}) = \mathscr{S}$ and $\ker P_{A,\mathscr{S}} = A^{-1}(\mathscr{S}^\perp)$. Therefore, (15) follows from Ptak formula (13). ∎

Remark 5.3. Let $A \in L(\mathscr{H})^+$ such that the pair $(A, \mathscr{S})$ is compatible and $\ker A \cap \mathscr{S} = \{0\}$. Then, if $P_{\ker A}$ is the orthogonal projection onto $\ker A$, then

$$\|P_{A,\mathscr{S}}\| \geqslant (1 - \|P_{\ker A} P\|^2)^{-1/2}.$$

Indeed, if Q is the projection of Eq. (15), then $P_{\ker A} \leqslant Q$ because $\ker A \subseteq A^{-1}(\mathscr{S}^\perp)$. Then $\|P_{\ker A} P\|^2 = \|P P_{\ker A} P\| \leqslant \|PQP\| = \|QP\|^2$. This inequality, shown by de Boor [4] in the closed range case, relates the norm of $P_{A,\mathscr{S}}$ with the angle between $\ker A$ and $\mathscr{S}$.

PROPOSITION 5.4. *Let $A \in L(\mathscr{H})^+$ and let $\mathscr{S}_n$ $(n \in \mathbb{N})$ be closed subspaces such that all pairs $(A, \mathscr{S}_n)$ are compatible. Denote $\mathscr{M}_n = \overline{A^{1/2}(\mathscr{S}_n)}$, $n \in \mathbb{N}$.*

1. *The following conditions are equivalent:* (a)$P_{A,\mathscr{S}_n} \to^{A-SOT} 0$. (b)$\langle AP_{A,\mathscr{S}_n}\xi, \xi\rangle \to 0$, *for every $\xi \in \mathscr{H}$ (i.e. $AP_{A,\mathscr{S}_n} \to^{WOT} 0$ by polarization).* (c)$AP_{A,\mathscr{S}_n} \to^{SOT} 0$. (d)$\Sigma(P_{\mathscr{S}_n}, A) \to^{SOT} A$. (e)$P_{\mathscr{M}_n} A^{1/2} \to^{SOT} 0$.
2. *If there exists $C \geqslant 0$ such that $\|P_{A,\mathscr{S}_n}\| \leqslant C$ for all $n \in \mathbb{N}$ and $P_{\mathscr{S}_n} A \to^{SOT} 0$, then $P_{A,\mathscr{S}_n} \to^{A-SOT} 0$.*
3. *If $P_{A,\mathscr{S}_n} \to^{A-SOT} 0$, then $P_{\mathscr{S}_n} A \to^{SOT} 0$.*

Proof.

1. Because $P^*_{A,\mathscr{S}_n} A = AP_{A,\mathscr{S}_n}$, it is clear that conditions (a)–(c) are equivalent. By Remark 4.4, $\Sigma(P_{\mathscr{S}_n}, A) = A(1 - P_{A,\mathscr{S}_n})$ so that (c) is equivalent

to (d). Finally, by Proposition 4.2, we know that $A^{1/2}P_{A,\mathcal{S}_n} = P_{\mathcal{M}_n}A^{1/2}$ and this shows that (a) is equivalent to (e).

2. Suppose that there exists $C \geqslant 0$ such that $\|P_{A,\mathcal{S}_n}\| \leqslant C$ for all $n \in \mathbb{N}$ and that $P_{\mathcal{S}_n}A \to^{SOT} 0$. Denote $P_n = P_{\mathcal{S}_n}$. The fact that $R(P_{A,\mathcal{S}_n}) = R(P_n)$ implies that $P_nP_{A,\mathcal{S}_n} = P_{A,\mathcal{S}_n}$. Therefore, for every $\xi \in \mathcal{H}$,

$$\|P^*_{A,\mathcal{S}_n}A\xi\| = \|P^*_{A,\mathcal{S}_n}P_nA\xi\| \to 0,$$

since $\|P_{A,\mathcal{S}_n}\|$ is bounded. Hence $P^*_{A,\mathcal{S}_n}A = AP_{A,\mathcal{S}_n} \to^{SOT} 0$ so that $P_{A,\mathcal{S}_n} \to^{A-SOT} 0$ by item 1.

3. Suppose that $P_{A,\mathcal{S}_n} \to^{A-SOT} 0$. Then, by item 1, $AP_{A,\mathcal{S}_n} \to^{SOT} 0$. Note that $P_{A,\mathcal{S}_n}P_n = P_n$, so that $P_nP^*_{A,\mathcal{S}_n} = P_n$. Given $\xi \in \mathcal{H}$, we have that

$$\|P_nA\xi\| = \|P_nP^*_{A,\mathcal{S}_n}A\xi\| = \|P_nAP_{A,\mathcal{S}_n}\xi\| \leqslant \|AP_{A,\mathcal{S}_n}\xi\| \to 0. \quad \blacksquare$$

Remark 5.5. With the notations of Proposition 5.4, it follows that $P_{A,\mathcal{S}_n} \to^{A-SOT} 0$ if and only if $A^{1/2}(1 - P_{A,\mathcal{S}_n})\xi \to A^{1/2}\xi$ for every $\xi \in \mathcal{H}$ or, equivalently, the spline interpolants $\xi_n = (1 - P_{A,\mathcal{S}_n})\xi$ satisfy that $T\xi_n \to T\xi$ in $\mathcal{H}_1$, if $T \in L(\mathcal{H}, \mathcal{H}_1)$ and $T^*T = A$. In particular, if $P_{A,\mathcal{S}_n} \to^{A-SOT} 0$, then $\min\{\|T(\xi + \tau)\| : \tau \in \mathcal{S}_n\} = \|T(1 - P_{A,\mathcal{S}_n})\xi\| \to \|T\xi\|$.

PROPOSITION 5.6. *Let $A \in L(\mathcal{H})^+$ and $\mathcal{S}_2 \subseteq \mathcal{S}_1 \subseteq \mathcal{H}$ be closed subspaces. Suppose that $(A, \mathcal{S}_1)$ is compatible. Denote by $P_i = P_{\mathcal{S}_i}$, $i = 1, 2$ and $a_1 = P_1AP_1 \in L(\mathcal{S}_1)^+$. Then*

$(A, \mathcal{S}_2)$ is compatible if and only if $(a_1, \mathcal{S}_2)$ is compatible in $L(\mathcal{S}_1)$.

Proof. We know that, if $A = \begin{pmatrix} a_1 & b_1 \\ b_1^* & c_1 \end{pmatrix}$, in the matrix decomposition induced by P_1, then $R(b_1) \subseteq R(a_1)$. Hence also $R(P_2b_1) \subseteq R(P_2a_1)$. If $a_1 = \begin{pmatrix} a_2 & b_2 \\ b_2^* & c_2 \end{pmatrix}$, using now the matrix decomposition induced by P_2, then $P_2a_1 = a_2 + b_2$ and $P_2A(1 - P_2) = b_2 + P_2b_1$. Hence,

$$R(P_2b_1) \subseteq R(P_2a_1) = R(a_2) + R(b_2) \text{ and } R(P_2A(1 - P_2)) = R(b_2) + R(P_2b_1).$$

Therefore, the pair $(A, \mathcal{S}_2)$ is compatible if and only if $R(P_2A(1 - P_2)) \subseteq R(P_2AP_2) = R(a_2)$ if and only if $R(b_2) \subseteq R(a_2)$ if and only if the pair $(a_1, \mathcal{S}_2)$ is compatible. $\blacksquare$

EXAMPLE 5.7. Let $A \in L(\mathcal{H})^+$ injective but not invertible. With the notations of Proposition 5.6 it is easy to see that $P_1P_{A,\mathcal{S}_2}P_1 = P_{A,\mathcal{S}_2}P_1 \in \mathcal{P}(a_1, \mathcal{S}_2)$. Note that a_1 is injective, so that $\mathcal{P}(a_1, \mathcal{S}_2)$ has a unique element and

$$P_{a_1,\mathcal{S}_2} = P_{A,\mathcal{S}_2}P_1 \Rightarrow \|P_{A,\mathcal{S}_2}\| \geqslant \|P_{a_1,\mathcal{S}_2}\|. \tag{16}$$

We shall see that there exists a sequence $\mathcal{S}_n$, $n \in \mathbb{N}$, of closed subspaces of $\mathcal{H}$ such that

1. the pair $(A, \mathcal{S}_n)$ is compatible for every $n \in \mathbb{N}$,
2. $\mathcal{S}_{n+1} \subseteq \mathcal{S}_n$ for every $n \in \mathbb{N}$,
3. $\bigcap_{n \geqslant 1} \mathcal{S}_n = \{0\}$, so that $P_{\mathcal{S}_n} \to^{SOT} 0$,
4. $\|P_{A,\mathcal{S}_n}\| \to \infty$.

In order to prove this fact, we need the following lemma:

LEMMA 5.8. *Let $B \in L(\mathcal{H})^+$ be injective non-invertible. Then, for every $\varepsilon > 0$, there exists a closed subspace $\mathcal{S} \subseteq \mathcal{H}$ such that the pair $(B, \mathcal{S})$ is compatible, $P_{\mathcal{S}}BP_{\mathcal{S}}$ is not invertible in $L(\mathcal{S})$ and $\|P_{B,\mathcal{S}}\| \geqslant \varepsilon^{-1}$.*

Proof. Let $\eta \in \mathcal{H}$ be a unit vector. Denote by $\xi = B\eta$ and consider the subspace $\mathcal{S} = \{\xi\}^\perp$ and $P = P_{\mathcal{S}}$. It is clear that $\eta \in B^{-1}(\mathcal{S}^\perp)$. First note that $\langle \xi, \eta \rangle = \langle B\eta, \eta \rangle > 0$, so that $\eta \notin \mathcal{S}$. Since $\mathcal{S}$ is an hyperplane, this implies that $\mathcal{S} + B^{-1}(\mathcal{S}^\perp) = \mathcal{H}$ and the pair $(B, \mathcal{S})$ is compatible. Also PBP is not invertible because $dim\, \mathcal{S}^\perp = 1 < \infty$. Note that $B^{-1}(\mathcal{S}^\perp)$ is the subspace generated by η. Hence, if $Q = P_{B^{-1}(\mathcal{S}^\perp)}$, it is easy to see that $\|PQ\| = \|P\eta\|$. Then, by Eq. (15),

$$\|P_{B,\mathcal{S}}\| = (1 - \|PQ\|^2)^{-1/2} = (1 - \|P\eta\|^2)^{-1/2} = \|(1-P)\eta\|^{-1}$$

and

$$\|(1-P)\eta\| = \left| \left\langle \eta, \frac{\xi}{\|\xi\|} \right\rangle \right| = \frac{\langle \eta, B\eta \rangle}{\|B\eta\|}.$$

So, it suffices to show that there exists a unit vector η such that $\langle \eta, B\eta \rangle \leqslant \varepsilon \|B\eta\|$. Consider $\rho \in \mathcal{H} \setminus R(B^{1/2})$ a unit vector. Let ρ_n be a sequence of unit vectors in $R(B^{1/2})$ such that $\rho_n \to \rho$. Let $\mu_n \in \mathcal{H}$ such that $B^{1/2}\mu_n = \rho_n$, $n \in \mathbb{N}$, and denote by $\xi_n = B^{1/2}\rho_n = B\mu_n$, and $\xi = B^{1/2}\rho$. It is easy to see, using that $B(\mu_n) = \xi_n \to \xi \notin R(B)$, that $\|\mu_n\| \to \infty$. Denote by $\eta_n = \mu_n\|\mu_n\|^{-1}$. Then

$$\frac{\langle \eta_n, B\eta_n \rangle}{\|B\eta_n\|} = \frac{\langle \mu_n, B\mu_n \rangle}{\|\mu_n\|^2 \|B\eta_n\|} = \frac{\|B^{1/2}\mu_n\|^2}{\|\mu_n\| \; \|B\mu_n\|} = \frac{1}{\|\mu_n\| \; \|\xi_n\|} \to 0$$

because $\xi_n \to \xi \neq 0$. ∎

By an inductive argument, using Lemma 5.8, Proposition 5.6 and Eq. (16), we can construct a sequence of compatible subspaces $\mathcal{S}_n$, $n \in \mathbb{N}$, such that $\mathcal{S}_{n+1} \subseteq \mathcal{S}_n$ and $\|P_{A,\mathcal{S}_n}\| \to \infty$. We can also get that $P_{\mathcal{S}_n} \to^{SOT} 0$ by interlacing, before constructing the subspace $\mathcal{S}_{n+1}$, a spectral subspace $\mathcal{T}_n$ of $P_{\mathcal{S}_n}AP_{\mathcal{S}_n}$ (as an operator in $L(\mathcal{S}_n)$), in such a way that $P_{\mathcal{T}_n}AP_{\mathcal{T}_n}$ is not invertible and the projections $P_{\mathcal{T}_n} \to^{SOT} 0$ (this can be done recursively by testing the projections $P_{\mathcal{T}_n}$ in the first n elements of a countable dense subset of $\mathcal{H}$), and taking $\mathcal{S}_{n+1}$ as a subspace of $\mathcal{T}_n$. Note that the pairs $(P_{\mathcal{S}_n}A P_{\mathcal{S}_n}, \mathcal{T}_n)$ are clearly compatible, so that also the pairs $(A, \mathcal{T}_n)$ are compatible by Proposition 5.6.

Remark 5.9. Recall from Remark 4.4 that if $(A, \mathcal{S})$ is compatible, then $A(1 - P_{A,\mathcal{S}}) = \Sigma(P, A)$. Then

$$0 \leqslant AP_{A,\mathcal{S}} = A - \Sigma(P, A) \leqslant A.$$

This implies that $\|AP_{A,\mathcal{S}}\| \leqslant \|A\|$, while $\|P_{A,\mathcal{S}}\|$ can be arbitrarily large.

6. SOME EXAMPLES

In this section, we present several examples of pairs $(A, \mathcal{S})$ which are not compatible and pairs $(A, \mathcal{S})$ which are compatible and such that the spline projector $P_{A,\mathcal{S}}$ can be explicitly computed. Observe that Example 6.4 cannot be studied under the closed range hypothesis, considered by Atteia, de Boor and Izumino.

EXAMPLE 6.1. Let $A \in L(\mathcal{H})^+$ and

$$M = \begin{pmatrix} A & A^{1/2} \\ A^{1/2} & I \end{pmatrix} = \begin{pmatrix} A^{1/2} & 0 \\ I & 0 \end{pmatrix} \begin{pmatrix} A^{1/2} & I \\ 0 & 0 \end{pmatrix} \in L(\mathcal{H} \oplus \mathcal{H})^+.$$

Denote by $\mathcal{S} = \mathcal{H} \oplus \{0\}$ and by $N = \begin{pmatrix} A^{1/2} & I \\ 0 & 0 \end{pmatrix}$. Since $M = N^*N$, then $ker\, M = ker\, N = \{\xi \oplus -A^{1/2}\xi \colon \xi \in \mathcal{H}\}$ which is the graph of $-A^{1/2}$. Note that $R(N) = (R(A^{1/2}) + R(I)) \oplus \{0\} = \mathcal{S}$, so that $R(M)$ is also closed. If A is injective with non-closed range, then $(M, \mathcal{S})$ is not compatible (because $R(A)$ is properly included in $R(A^{1/2})$). Observe that this implies that the inclination between $\mathcal{S}$ and $ker\, M$ is one, cf. [4].

Remark 6.2. Let $P \in \mathcal{P}$, $R(P) = \mathcal{S}$ and $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \in L(\mathcal{H})^+$. It is well known that the positivity of A implies that $R(b) \subseteq R(a^{1/2})$. Therefore, if $dim\, \mathcal{S} < \infty$ then the pair $(A, \mathcal{S})$ is compatible : in fact in this case $R(a) = R(PAP)$ must be closed, so $R(b) \subseteq R(a^{1/2}) = R(a)$ and Corollary 2.2, can be applied. On the other hand, if $dim\,\mathcal{S}^{\perp} < \infty$ and $R(A)$ is closed then, by Remark 2.4, $(A, \mathcal{S})$ is compatible. However, if $R(A)$ is not closed, then the pair $(A, \mathcal{S})$ can be non-compatible:

PROPOSITION 6.3. *Let $P \in \mathcal{P}$, $R(P) = \mathcal{S}$ and $A \in L(\mathcal{H})^{+}$. Suppose that A is injective non-invertible and $dim\,\mathcal{S}^{\perp} < \infty$. Then $(A, \mathcal{S})$ is compatible if and only if $\mathcal{S}^{\perp} \subseteq R(A)$.*

Proof. By Eq. (2), $(A, \mathcal{S})$ is compatible if and only if $A^{-1}(\mathcal{S}^{\perp}) + \mathcal{S} = \mathcal{H}$. Since A is injective, Eq. (3) says that $A^{-1}(\mathcal{S}^{\perp}) \cap \mathcal{S} = \{0\}$. Now the result becomes clear because $dim\,A^{-1}(\mathcal{S}^{\perp}) = dim\,(\mathcal{S}^{\perp} \cap R(A))$. ∎

EXAMPLE 6.4. Let $T \in L(\mathcal{H}, L^2)$ given by $Te_m = \frac{e^{i(m+1)t}}{m}$, where e_m $(m \in \mathbb{N})$ is an orthonormal basis of $\mathcal{H}$. Then $A = T^*T$ is given by $Ae_m = \frac{e_m}{m^2}$, which is injective non-invertible. Let $\xi_1, \ldots, \xi_n \in R(A)$, denote by $\mathcal{S} = \{\xi_1, \ldots, \xi_n\}^{\perp}$ and $P = P_{\mathcal{S}}$. If $\xi_i = (\xi_i^{(1)}, \xi_i^{(2)}, \ldots, \xi_i^{(m)}, \ldots)$, $1 \leqslant i \leqslant n$, denote by

$$\eta_i = (\xi_i^{(1)}, 4\xi_i^{(2)}, \ldots, m^2\xi_i^{(m)}, \ldots) \in \mathcal{H}, \quad 1 \leqslant i \leqslant n,$$

and Q the orthogonal projection onto the subspace $\mathcal{T}$ generated by $\eta_1, \ldots, \eta_n$. It is clear that $\mathcal{T} = A^{-1}(\mathcal{S}^{\perp})$. Then $(A, \mathcal{S})$ is compatible and $P_{A,\mathcal{S}}$ is the projection onto $\mathcal{S}$ with kernel $\mathcal{T}$. Therefore (cf. [5] or [17]), $\|PQ\| < 1$,

$$P_{A,\mathcal{S}} = (1 - QP)^{-1}(1 - Q) = \sum\nolimits_{k=0}^{\infty} (QP)^k (1 - Q)$$

and $\|P_{A,\mathcal{S}}\| = \|1 - P_{A,\mathcal{S}}\| = (1 - \|PQ\|^2)^{-1/2}$.

Remark 6.5. Let $B \in L(\mathcal{H})^{+}$ be injective and non-invertible. Let $\xi \in \mathcal{H}$ be a unit vector, $\mathcal{S} = \{\xi\}^{\perp}$, $P = P_{\mathcal{S}}$ and $P_{\xi} = 1 - P$. Let $B = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ in terms of P. By Proposition 6.3, $(B, \mathcal{S})$ is compatible if and only if $\xi \in R(B)$. Note that the sequence ξ_n (in $R(B)$) of Lemma 5.8 converges to $\xi \notin R(B)$. This is, precisely, the fact which implies that $\|P_{B,\{\xi_n\}^{\perp}}\|$ converges to infinity.

EXAMPLE 6.6. Fix $\mathcal{S}$ a closed subspace of $\mathcal{H}$ and consider the set

$$\mathcal{A}_{\mathcal{S}} = \{A \in L(\mathcal{H})^{+}\colon \text{ the pair } (A, \mathcal{S}) \text{ is compatible}\}$$

and the map $\alpha\colon \mathcal{A}_{\mathcal{S}} \to \mathcal{Q}$ given by $\alpha(A) = P_{A,\mathcal{S}}$. We shall see that α is not continuous. Indeed, let $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$, and suppose that $R(b) = R(a)$ is a closed subspace $\mathcal{M}$ properly included in $\mathcal{S}$. Denote by $\mathcal{N} = \mathcal{S} \ominus \mathcal{M}$ and consider the projection $P_{\mathcal{N}}$ and some element $u \in L(\mathcal{S}^{\perp}, \mathcal{N}) \subseteq L(\mathcal{H})$, $u \neq 0$. Consider, for every $n \in \mathbb{N}$,

$$A_n = A + \frac{1}{n}(P_{\mathcal{N}} + u)^*(P_{\mathcal{N}} + u) = A + \frac{1}{n} = \begin{array}{cc} \begin{pmatrix} 1 & 0 & u \\ 0 & 0 & 0 \\ u^* & 0 & u^*u \end{pmatrix} & \begin{array}{l} \mathcal{N} \\ \mathcal{M} \\ \mathcal{S}^{\perp} \end{array} \end{array}$$

$$= \begin{pmatrix} \frac{1}{n} & 0 & \frac{1}{n}\,u \\ 0 & a & b \\ \frac{1}{n}\,u^* & b^* & c + \frac{1}{n}\,u^*u \end{pmatrix} \geqslant A \geqslant 0.$$

It is clear that $A_n \to A$. Note that a is invertible in $L(\mathcal{M})$. Then, by Theorem 2.3,

$$P_{A,\mathcal{S}} = \begin{array}{cc} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a^{-1}b \\ 0 & 0 & 0 \end{pmatrix} & \begin{array}{l} \mathcal{N} \\ \mathcal{M} \\ \mathcal{S}^{\perp} \end{array} \end{array},$$

Also $a + \frac{1}{n}P_{\mathcal{N}}$ is invertible in $L(\mathcal{S})$ for every $n \in \mathbb{N}$. Then,

$$P_{A_n,\mathcal{S}} = \begin{pmatrix} n & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{n} & 0 & \frac{1}{n}\,u \\ 0 & a & b \\ \frac{1}{n}\,u^* & b^* & c + \frac{1}{n}\,u^*u \end{pmatrix}$$

$$= \begin{array}{cc} \begin{pmatrix} 1 & 0 & u \\ 0 & 1 & a^{-1}b \\ 0 & 0 & 0 \end{pmatrix} & \begin{array}{l} \mathcal{N} \\ \mathcal{M} \\ \mathcal{S}^{\perp} \end{array} \end{array}$$

for all $n \in \mathbb{N}$. Therefore, $\alpha(A_n) = P_{A_n,\mathcal{S}} P_{A,\mathcal{S}} = \alpha(A)$. Note that the sequence $\alpha(A_n)$ converges (actually, it is constant) to $P_{A,\mathcal{S}} + u$, which belongs to $\mathcal{P}(A, \mathcal{S})$ by Theorem 2.3.

REFERENCES

1. W. N. Anderson and G. E. Trapp, Shorted operators II, *SIAM J. Appl. Math.* **28** (1975), 60–71.
2. E. Andruchow, G. Corach, and D. Stojanoff, Geometry of oblique projections, *Studia Math.* **137** (1999) 61–79.
3. M. Atteia, Generalization de la définition et des propriétés des "spline-fonctions," *C.R. Acad. Sci. París* **260** (1965), 3550–3553.
4. C. de Boor, Convergence of abstract splines, *J. Approx. Theory* **31** (1981), 80–89.
5. D. Buckholtz, Hilbert space idempotents and involutions, *Proc. Amer. Math. Soc.* **128** (2000), 1415–1418.
6. G. Corach, A. Maestripieri, and D. Stojanoff, Generalized Schur complements and oblique projections, *Linear Algebra Appl.* **341** (2002), 259–272.

7. G. Corach, A. Maestripieri, and D. Stojanoff, Oblique projections and Schur complements, *Acta Sci. Math.* (*Szeged*) **67** (2001) 337–356.
8. F. Deutsch, The angle between subspaces in Hilbert space, *in* "Approximation Theory, Wavelets and Applications" (S. P. Singh, Ed.), pp. 107–130, Kluwer, Netherlands, 1995.
9. F. J. Delvos, Splines and pseudoinverses, *RAIRO Anal. Numér.* **12** (1978), 313–324.
10. R. G. Douglas, On majorization, factorization and range inclusion of operators in Hilbert space, *Proc. Amer. Math. Soc.* **17** (1966) 413–416.
11. M. Golomb, "Splines, *n*-Widths and Optimal Approximations," MRC Technical Summary Report 784, 1967.
12. S. Hassi, and K. Nordström, On projections in a space with an indefinite metric, *Linear Algebra Appl.* **208/209** (1994), 401–417.
13. S. Izumino, Convergence of generalized splines and spline projectors, *J. Approx. Theory* **38** (1983), 269–278.
14. M. G. Krein, The theory of self-adjoint extensions of semibounded Hermitian operators and its applications, *Mat. Sb.* (*N.S.*) **20** (62) (1947), 431–495.
15. Z. Pasternak-Winiarski, On the dependence of the orthogonal projector on deformations of the scalar product, *Studia Math.* **128** (1998), 1–17.
16. E. L. Pekarev, Shorts of operators and some extremal problems, *Acta Sci. Math.* (*Szeged*) **56** (1992), 147–163.
17. V. Ptak, Extremal operators and oblique projections, *Casopis Pest. Mat.* **110** (1985), 343–350.
18. A. Sard, Optimal approximation, *J. Funct. Anal.* **1** (1967), 222–244.
19. B. Shekhtman, Unconditional convergence of abstract splines, *J. Approx. Theory* **30** (1980), 237–246.